Biangular and bipolar coordinates defining plane curves
Sectrices of Maclaurin / Plateau's curves
- Part XXIV -

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Abstract

Some curves have a simple definition in a biangular system of coordinates. A point A in the plane is located by a system of two lines rotating around two fixed points B, C on an xx'-axis and two angles (θ, θ'). Triangle ABC defines a biangular coordinate system if the two angles depends on one parameter. A similar system is the bipolar coordinates (well known for central conics and their foci), with distances (ρ, ρ') to the poles (B, C) fixed on axis-x'x and a relation between the two lengths. We examine a few examples of the Maclaurin sectrices or Plateau's curves.

1 Biangular and bipolar coordinates

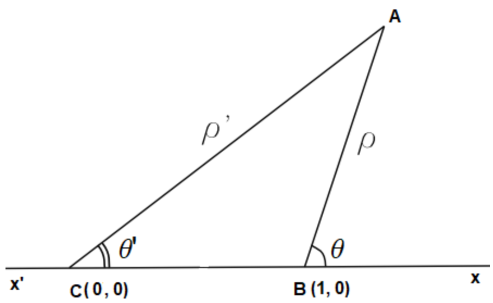

Figure 1: Bipolar (ρ, ρ') and biangular (θ, θ') coordinates with poles C, B

Biangular coordinates use two fixed points C(0, 0) and B(1, 0) and two lines rotating around these points. The two angles are measured from the direction x'x defined by the two points B and C on this axis. The position of point A is the third vertice of the triangle CAB. When the lines intersect the common point is A. When they are parallele the point A goes to infinity : it is an asymptotic direction.
The coordinates are two angles $\theta = \angle$ xBA and $\theta' = \angle$ xCA. When $\theta = \theta' + k\pi$ we get the asymptotic directions of the curve. The biangular equation of a curve is a relation between these angles f(θ, θ')=0 or θ'=$g(\theta)$ or $\theta = h_1(t)$ and $\theta' = h_2(t)$ (with t a parameter).
The bipolar coordinates of A, with same poles C and B, are two lengths : $\rho =$ BA and ρ' = CA. Here the condition for the existence of point A is the triangular inequality: $\rho + \rho' \geq BC = 1$.
Bipolar coordinates have connections with the biangular ones. We shall often use the sine formula in the triangle with usual notations - see fig. (2) -:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{abc}{2.S} = 2.R$$

where S is the area and R the radius of the circumcircle.

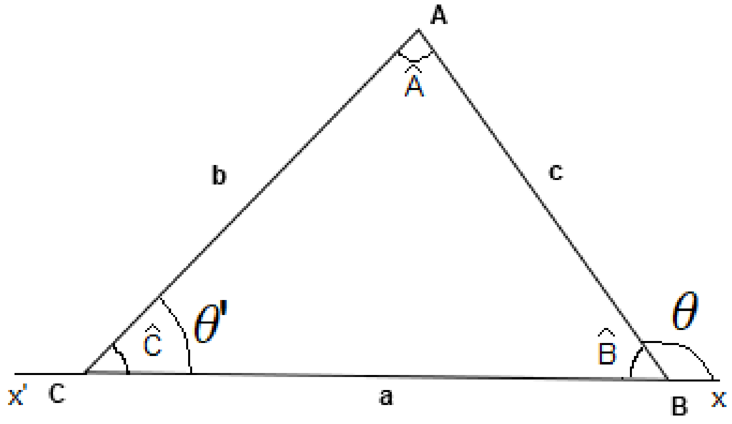

Figure 2: A triangle : vertices A, B, C and sides a, b, c.

1.1 Orthogonal trajectories of generalized cassinians. Sectrices of Maclaurin.

Given a biangular or bipolar equation of a family of plane curves, it is possible to find orthogonal trajectories - see (1). If a family of curves,locus of A, in bipolar is : $f(b, c) = f(\rho, \rho') = h$, with h the parameter, then :

$$\frac{\partial f}{\partial \rho}.d\rho + \frac{\partial f}{\partial \rho'}.d\rho' = 0$$

to find the orthogonal trajectories of this family of curves we change in this equation the ratio $d\rho/d\rho'$ by the ratio $\rho.d\theta/\rho'.d\theta'$. So for orthogonal trajectories we have :

$$\frac{\partial f}{\partial \rho}.\rho.d\theta + \frac{\partial f}{\partial .\rho'}\rho'.d\theta' = 0$$

In the triangle ABC we have :

$$\frac{\rho}{\rho'} = \frac{\sin\theta}{\sin\theta'}$$

We can eliminate the ratio ρ/ρ' between the above equations and the resulting differential equation is the one of orthogonal trajectories in biangular coordinates this time.
For the general Cassinian ovals $f = \rho^n.\rho'^p$ =constant, then

$$n.\rho^{n-1}\rho'^p.d\rho + p.\rho^n.\rho'^{p-1}.d\rho' = 0$$

$$n.\rho^n\rho'^p.d\theta + p.\rho^n\rho'^p.d\theta' = 0$$

$$n.d\theta + p.d\theta' = 0$$

and so :

$$n.\theta + p.\theta' = \mathrm{K} = \text{constant}$$

This last formula is the the biangular expression for the Maclaurin sectrices (a linear relation, if n, p are rational numbers, between θ and θ').

2 Curves defined by biangular coordinates with $\theta' = f(\theta)$.

We suppose the fixed points are B and C on the x'x-axis. The angle $B = \theta$ and the second angle $C = \theta' = f(\theta)$ - so angle $B = f^{-1}(\theta')$ - Then angle $A = \pi - \theta - f(\theta)$. In the triangle

ABC the relations between sides and angles are :

$$\frac{\rho}{\sin\theta'} = \frac{\rho'}{\sin(\pi-\theta)} = \frac{a}{\sin\widehat{A}} = \frac{a}{\sin(\theta-\theta')}$$

from these equations and the angles relations we deduce two (mono)-polar coordinates equations for each fixed point on the x'x-axis. The equations of these curves in polar coordinates (ρ,θ) or (ρ',θ') and $\theta' = f(\theta)$ - with condition $f(\theta) \neq \theta$ - are :

$$\boxed{\rho = a.\frac{\sin[f(\theta)]}{\sin[f(\theta)-\theta]}} \text{ pole at B .}$$

$$\boxed{\rho' = a.\frac{\sin(f^{-1}(\theta'))}{\sin[f^{-1}(\theta')-\theta']}} \text{ pole at C.}$$

3 Sectrices of Maclaurin

The sectrices of Maclaurin, with angular offset, are defined by $\theta' = f(\theta) = k.\theta + \theta_o$ with condition $(k \neq 1)$:

$$\boxed{\rho = a.\frac{\sin[k.\theta+\theta_0)]}{\sin[(1-k)\theta+\theta_0]}} \text{ pole at B.}$$

If $\theta_o = 0$ the polar equation is :

$$\boxed{\rho = a.\frac{\sin[k.\theta)]}{\sin[(1-k)\theta]}} \text{ pole at B.}$$

If k is changed in -k the equation become with change of pole :

$$\boxed{\rho = a.\frac{\sin[k.\theta)]}{\sin[(k+1)\theta]}} \text{ pole at C.}$$

A similar derivation with the other pole C gives and $k.\theta = \phi - \phi_o$ gives :

$$\boxed{\rho' = a.\frac{\sin[(\phi-\phi_0)/k]}{\sin[(\phi-\phi_0)/k-\phi]}} \text{ pole at C.}$$

If $\phi_o = 0$ the polar equation is :

$$\boxed{\rho' = a.\frac{\sin(\phi/k)}{\sin[(k-1)/k)\phi]}} \text{ pole at C.}$$

3.1 Plateau's curves

If we choose commensurables values of k ($k = m/n, m \neq n$ and m, n integers) so that curves are algebraic then for $\theta_o = 0$ and $\phi_o = 0$ the first above equation can be transformed in the following parametric equations - the origin of orthonormal coordinates is now at $(a/2, 0)$ - :

$$\boxed{x = a.\frac{\sin[(m+n)t]}{2\sin[(m-n)t]} \text{ and } y = a.\frac{\sin(mt).\sin(nt)}{\sin[(m-n)t]}}$$

3.2 Bipolar and biangular coordinates, inversions with centers at the poles and an axial symmetry.

The poles B and C that define the curve and inversions centered at these poles are useful in the geometry of the sectrices. We trace the circles tangent at B and C to the base line x'x and passing through A. We impose that BC is a constant length (=1 w.l.o.g.) and point A is moving on any curve in the plane then the points B' and C' second point of intersection of BA and CA respectively with the circles angent at B and C to x'x (see figure) then B' and C' will move on the inverses of A-curve w.r.t. centers B and C respectively. If point I is the middle of BC the second point of intersection of tangent circles to x'x called A' is the inverse of A-curve in the inversion: (I, $IB^2 = IC^2$).

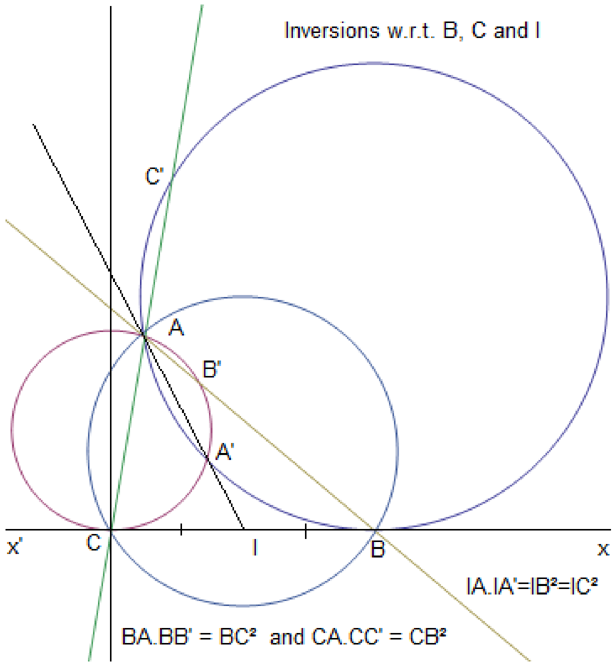

Figure 3: Inversions with centers at pole and circles through A tangent at B and C to x'x

A is a point moving on the curve in bipolar system of coordinates with poles in C and B on x'x axis. B' and C' are inverses of A as above. We call B" and C" the inverses of B' (C, CB^2) and C' (B, BC^2). The 5 points A, B', B", C', C" are cocyclic. It is possible to define inversions that transform the locus of A. For these inversions we choose the poles at C or B and the radius of inversion equal to BC (=1).

We can identify 3 inversions and an axial symmetry listed in the following table :

No	Inversions or symmetry	Pole / axis
1	$BA.BB' = BC^2$	B
2	$BC'.BC" = BC^2$	B
3	$CA.CC' = CB^2$	C
4	$CB'.CB" = CB^2$	C
5	B" Sym C' and B' sym C"	Axis TI (mediator of CB)
6	$IA.IA' = IC^2 = IB^2$	I

These transformations reduce the numbers of different curves if we distinguish loci up to an axial symmetry w.r.t. the mediator of BC.

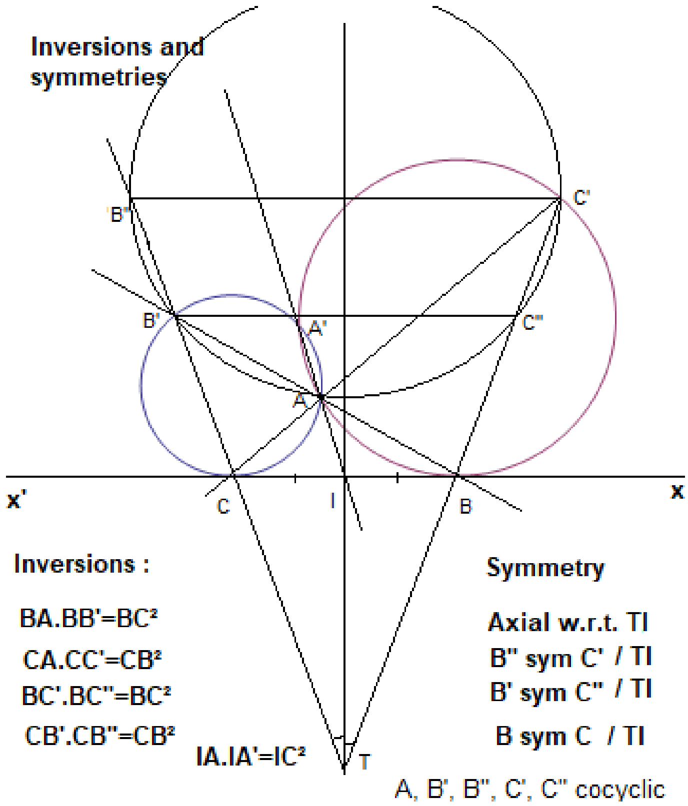

Figure 4: Inversions/symmetry - $\widehat{CBC'} = \widehat{BCB'} = \widehat{BAC}$ - points A, B', B", C', C" are cocyclic.

3.3 Triangles with linear angular relations. The number theory problem.

We can find many papers - see (1), (2), (3) - about commensurables triangles ABC with angular relations between two angles. For example $\widehat{C} = 2.\widehat{B}$, equivalent to a relation between the 3 angles since in euclidean geometry $\widehat{A}+\widehat{B}+\widehat{C} = \pi$. A linear relation between angles (A, B, C) - with rational coefficients - in the triangle. Using sine formulas a relation between angles can be translated, into a relation between the side lengths (a, b, c). We suppose the triangles on BC moves maintaining the same proportionality between angles. In paper (1) a few angle relations and the corresponding algebraic formulas relating the sides are proved and listed.
This problem is not strictly a geometric question since it focuses on the case of integer side lengths. So we can consider "integers" triangles as a generalization of Pythagorean triples. We limit here only a geometric point of view. Pythagorean triples are associated to the circle and relation $a^2 + b^2 = c^2$ and angular equivalent : $\widehat{A}+\widehat{B} = \widehat{C} = \pi/2$.
We recall some of the relations in the following table (see (1) for delails) :

No	$F(A,B,C)=0$	$f(a,b,c)=0$
1	$\widehat{A}=\widehat{B}$	$a=b$(isocele triangle)
2	$\widehat{A}+\widehat{B}=\widehat{C}$	$a^2+b^2=c^2$(rectangle triangle)
3	$\widehat{C}=2.\widehat{B}$	$c^2=b^2+ab$
4	$\widehat{C}=2.\widehat{B}+\widehat{A}$	$c^2=b^2+ac$
5	$\widehat{C}=2.\widehat{A}-\widehat{B}$	$a^2=b^2+c^2-bc$
6	$\widehat{C}=(1/2)\widehat{A}-\widehat{B}$	$a^2=b^2+c^2+bc$
7	$\widehat{C}=2.(\widehat{B}-\widehat{A})$	$bc^2=(b-a)(b+a)^2$
8	$\widehat{B}=3.\widehat{C}$	$a^2c=(b+c)(b-c)^2$

If we fix one of the three side, say a, and put it on x'x coinciding with length CB=1 then the last column of the above table gives the corresponding bipolar equation equation $f(1,b,c)=0$. If we put b or c on x'x gives the two other bipolar equations : $f(a,1,c)=0$ and $f(a,b,1)=0$.

3.4 Triangles with linear angular relations. Example : $\widehat{C}=k.\widehat{B}$. k=constant

These curves are defined by mean of a triangle with two fixed points C and B on the x'x-axis repecting a linear relation between two angles in the triangle and called sectrices of Maclaurin. The triangle depends on a parameter θ, so the it is defined by one parameter and the third top A describes a plane curve given by a biangular equation. We can place the triangle in three possible positions on the pole-axis : AB, AC in place of BC on the base line (x'x-axis) when the linear relation between angles is held during the movement.

3.5 The simplest example : $\widehat{C}=\widehat{B}$, the isoceles triangle

In the case where angles $\widehat{C}=\widehat{B}$ are on the x'x-axis, then the third vertice A is on the mediator of BC. The locus of A is a line. The relation between sides is $c=b$.
If A and B are fixed ($\widehat{C}=\widehat{B}$) then AC is constant (=1) so the point C is on a circle centered in A. Note that these circles and lines are exchanged by inversion. In the same way if A and C are fixed ($\widehat{C}=\widehat{B}$) then AB is constant (=1) so the point B is on a circle centered in A.

3.6 Another example : $\widehat{C}=2.\widehat{B}$

In this case $\widehat{C}=2.\widehat{B}$, there is a relation between the sides of the triangle : $c^2=b^2+ab$.
1- If D is the foot on AB of bissectrix CD of angle $\widehat{C}$ then a well known theorem gives $AD=\frac{b}{b+a}c$.
2- The triangles ADC and ACB are similar (see fig. 5-1) so :

$$\frac{AD}{b}=\frac{b}{c}\longrightarrow AD=\frac{b^2}{c}=\frac{b}{b+a}.c\longrightarrow c^2=b(b+a).$$

$$\frac{\rho}{\sin 2.\theta}=\frac{\rho'}{\sin(\pi-\theta)}=\frac{a}{\sin\widehat{A}}=\frac{a}{\sin(\theta-\theta')}$$

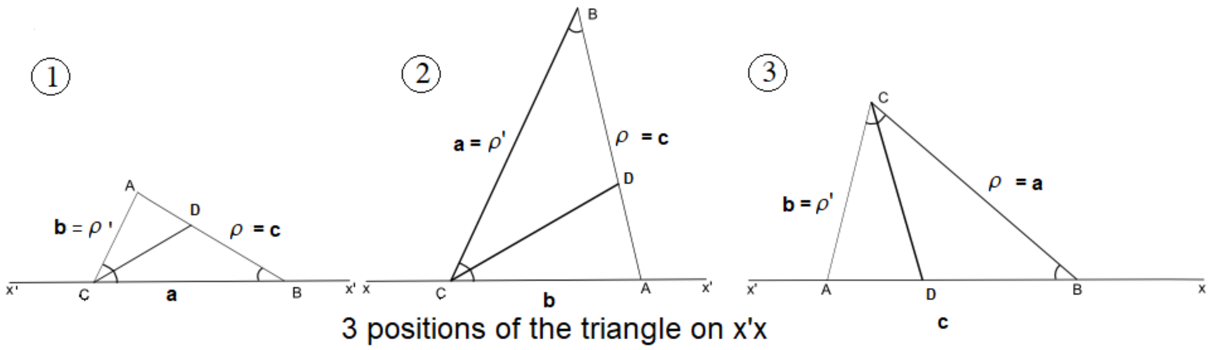

Figure 5: $\widehat{C}=2.\widehat{B}$ and the 3 positions of the triangle on x'x

3.6.1 $\widehat{C}=2.\widehat{B}$ and vertices C, B on x'x-axis

Since $c^2 = b^2 + ab$ we can write c^2 in two manners (CB on x'x see fig. 5-1), and then calculate b, if ϕ is angle C, we have :

$$c^2 = a^2 + b^2 - 2ab\cos\theta \text{ and } c^2 = b^2 + ab$$

$$a^2 - ab(1+2\cos\phi) = 0 \qquad \text{so :} \qquad b = \rho' = \frac{a}{1+2\cos\phi}$$

This is the polar equation of a conic $\rho = \frac{p}{1+e.\cos\theta}$ with the pole at a focus. Since e=2, our conic is a hyperbola of excentricty 2.
Now we evaluate $b^2 = \rho^2$, (angle B is $\theta = \phi/2$) :

$$c^2 = a^2 + \frac{a^2}{(1+2\cos\phi)^2} - \frac{2a^2\cos\phi}{1+2\cos\phi} = a^2\frac{2(1+\cos\phi)}{(1+2\cos\phi)^2} = a^2\frac{4\cos^2(\phi/2)}{(1+2\cos\phi)^2}$$

so $\rho = c = a\frac{2\cos(\phi/2)}{4\cos^2(\phi/2)-1} = a\frac{2\cos\theta}{4\cos^2\theta-1}$, this is the polar equation of an hyperbola of excentricity e=2. But this time the pole is on the summit on the other branch of the hyperbola.

3.6.2 $\widehat{C}=2.\widehat{B}$ and C, A on x'x-axis

The sine formula (see fig. 5-2) gives :

$$\frac{\rho'}{\sin 3\theta} = \frac{\rho}{\sin\theta} = \frac{a}{\sin 2\theta}$$

And $\rho' = a\frac{\sin 3\theta}{\sin 2.\theta} = a.\frac{4\cos^2\theta-1}{2\cos\theta}$. this polar equation is the one of the trissetrix of Maclaurin inverse of the above hyperbola. The pole is at the double point.
We have also :$\rho = \frac{a\sin\theta}{\sin 2\theta} = \frac{a}{2\cos\theta}$. Since the polar angle $\phi = 3\theta$ then the polar equation is $\rho = \frac{a}{2.\cos\phi/3}$. This is the equation represent also the Trissectrix if Maclaurin, the pole is this time at the summit on the symmetry axis.

3.6.3 $\widehat{C}=2.\widehat{B}$ and A, B on x'x-axis

The sine formula (applied on fig. 5-3) gives :

$$\frac{\rho'}{\sin 3\theta} = \frac{\rho}{\sin 2\theta} = \frac{a}{\sin\theta}$$

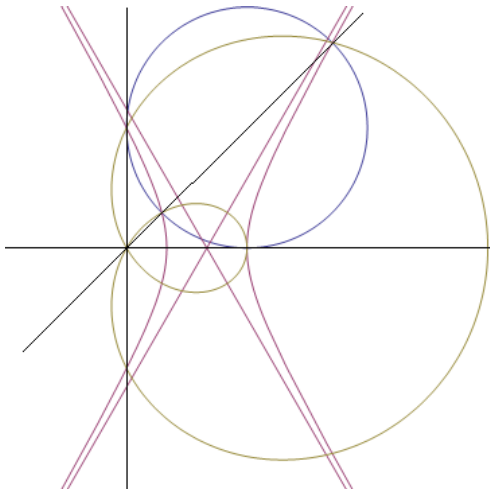

Figure 6: The Hyperbola (e=2) and Pascal Limacon

Then $\rho' = a\frac{\sin 3\theta}{\sin\theta} = a(4\cos^2\theta - 1)$. And since $\phi = 2\theta$ then $\rho' = a(4\cos^2\phi/2 - 1) = a(1 + 2\cos\phi)$.
This curve is a Limacon of Pascal with pole at the double point and inverse of the hyperbola with excentricity=2.
Similarly $\rho = a\frac{\sin 2\theta}{\sin\theta} = 2a.\cos\theta$. And since $\phi = 3.\theta$ then $\rho = 2a.\cos\phi/3$.
this is the equation of a Limacon of Pascal with pole at the summit of the little loop.
So we have the triples of curves in the plane :

- An hyperbola of eccentricity e=2
- A trissectrix of Maclaurin
- A Limacon of Pascal.

These three curves, by pair, are transformed by inversion and possess linked properties. The three curves are exchanged by a group of inversion+reflection of six elements isomorphic to the permutation group S_3 the same that permutes the roots of the third degree equation or the group of geometric transformations that keeps vertices of an equilateral triangle.

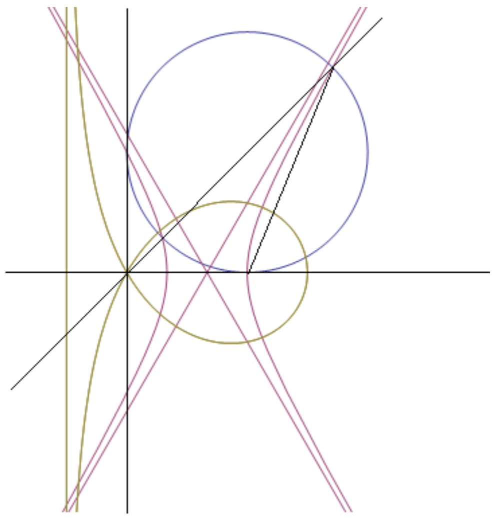

Figure 7: The Hyperbola (e=2) and Trisectrix of Maclaurin

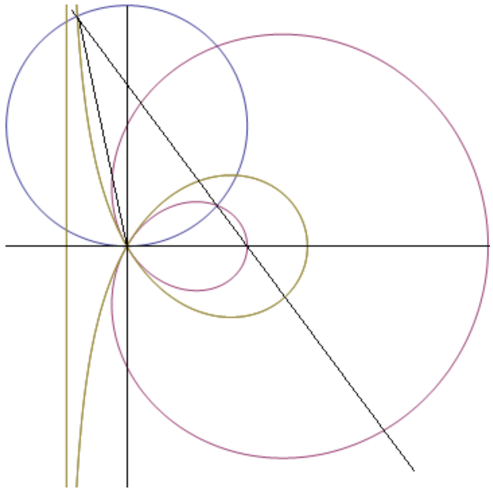

Figure 8: Trisectrix of Maclaurin and Pascal Limacon

4 The general case : $\widehat{C}$ = k . $\widehat{B}$ (k a rational parameter)

The sine formula in the triangle is : $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$. The two relations for C the external angle :

$$\widehat{C} = \phi = k.\theta = k.\widehat{B} \text{ and } \widehat{A} = \alpha = \phi - \theta$$

$$\frac{\sin \alpha}{a} = \frac{\sin \theta}{b} = \frac{\sin \phi}{c}$$

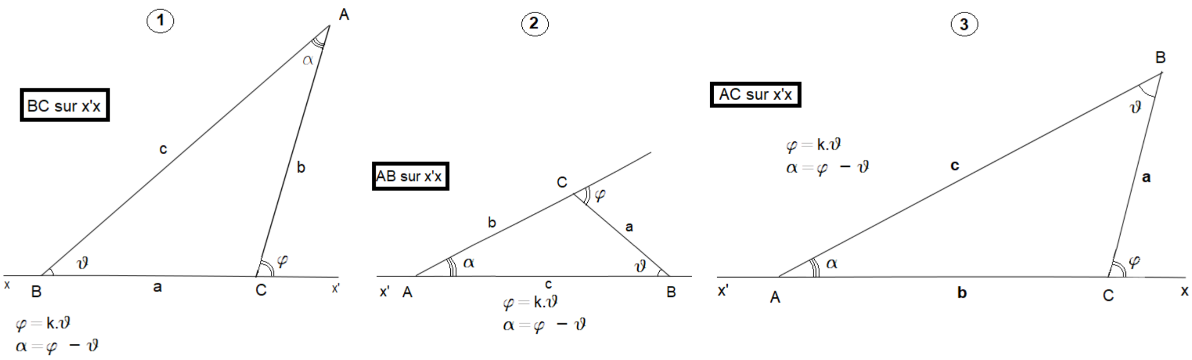

Figure 9: The general case $\widehat{C}$ = k . $\widehat{B}$ and the 3 positions of the triangle on x'x

4.1 1-Side BC on x'x

With the pole in C [BC on x'x] :

$$\rho = c = a\frac{\sin \phi}{\sin(\phi - \theta)} = a\frac{\sin k.\theta}{\sin(k-1)\theta}$$

With the pole in B [BC on x'x] :

$$\rho = b = a\frac{\sin \theta}{\sin(\phi - \theta)} = a\frac{\sin \phi/k.}{\sin \frac{k-1}{k}\phi}$$

4.2 2-Side AB on x'x

With the pole in B [AB on x'x] :

$$\rho = a = c\frac{\sin(\phi-\theta)}{\sin\theta} = a\frac{\sin(k-1)\theta}{\sin k.\theta}$$

With the pole in A [AB on x'x] :

$$\rho = b = c\frac{\sin\theta}{\sin\phi} = c\frac{\sin\frac{1}{k-1}\alpha}{\sin\frac{k}{k-1}\alpha}$$

4.3 3-Side AC on x'x

With the pole in A [AC on x'x] we have $\alpha = \phi - \theta$ and :

$$\rho = c = b\frac{\sin\phi}{\sin\theta} = b\frac{\sin\frac{k}{k-1}\alpha}{\sin\frac{1}{k-1}\alpha}$$

With the pole in C [AC on x'x] :

$$\rho = a = b\frac{\sin(\phi-\theta)}{\sin\theta} = b\frac{\sin(\phi-\phi/k)}{\sin\phi/k} = b\frac{\sin\frac{k-1}{k}\phi}{\sin\phi/k}$$

These six equations represent only three curves because each of the above have polar equations w.r.t. two pole on the x'x axis (respectively in B, C or A).
We can resume these results in the following table :

Side on x'x	Left-pole	Right-pole
(BC)	$c = a\frac{\sin k.\theta}{\sin(k-1)\theta}$	$b = a\frac{\sin\phi/k.}{\sin\frac{k-1}{k}\phi}$
(AB)	$a = c\frac{\sin(k-1)\theta}{\sin k.\theta}$	$b = c\frac{\sin\frac{1}{k-1}\alpha}{\sin\frac{k}{k-1}\alpha}$
(AC)	$c = b\frac{\sin\frac{k}{k-1}\alpha}{\sin\frac{1}{k-1}\alpha}$	$a = b\frac{\sin\frac{k-1}{k}\phi}{\sin\phi/k}$

As we can see in the above table, the six polar equations can be paired and this shows that these equations give three couples of inverse curves.

5 Another method to create curves in bipolar/biangular coordinates

We consider a circle tangent at A to y-axis, and point O opposite to A on it (OA=1). A line (Δ) turn around A and cut the circle at an other point H. A line through O cuts line AH at M. We note $\theta = \widehat{AOM}$, $\rho = OM$ and search for the locus of M when angle $\widehat{AOM} = k.\widehat{AOH} = k.\widehat{yAM}$. We have the relation $OM = \rho.\cos(k-1)\phi = \cos\phi$ and so for pole at O :

$$\boxed{\rho = \frac{\cos\phi}{\cos(k-1)\phi} = \frac{\cos\theta/k}{\cos\frac{k-1}{k}\theta}}.$$

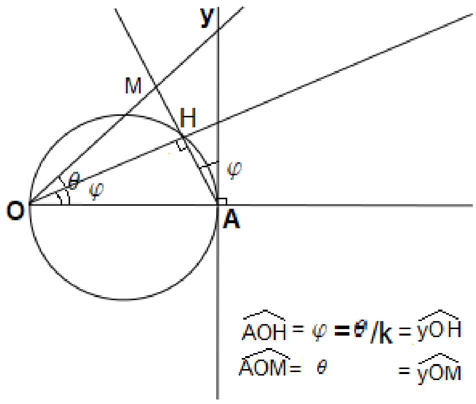

Figure 10: H sur le cercle of diameter AO, M on the curve

This family of curves depending on k contains some well known examples. The values k and k' linked by the involution $k.k' = k+k'$ or $k' = \frac{k}{k-1}$ and give two curves corresponding in the inversion centered in A.
This class-equation contains some known curves as for :

- $k' = -1 = \frac{k}{k-1}$ when k=1/2 the curve is the rectangular hyperbola $\rightarrow \rho = \frac{\cos\theta}{\cos 2\theta}$.
- $k = 1/2$ then curve C_k is the strophoid $\rightarrow \rho = \frac{\cos 2\theta}{\cos\theta}$
- $k = 2/3$ then curve C_k is the special limacon of Pascal $\rho = 2\cos\theta - 1$
- k=1 is the circle of diameter OA, the locus of H.
- k=2 Circle of center A.
- $k' = -2 = \frac{k}{k-1}$ when k=2/3 the curve is the hyperbola of excentricity e=2 with polar equation $\rho = \frac{1}{2\cos\theta - 1}$. This hyperbola and limacon of Pascal are inverse one of the other.

6 The general case : $\widehat{AOM}$ = k.$\widehat{AOH}$ or $\theta = k.\phi$

It is possible, by analogy with the case $\widehat{B} = k.\widehat{C}$ of the sections above, to place in turn each side of the triangle on the base-axis x'x. The case studied in preceeding section is the side OA on x'x. And we have found two polar equations with poles at O and at A. We shall examine the two other cases : side AM on x'x and side MO on x'x. The sine

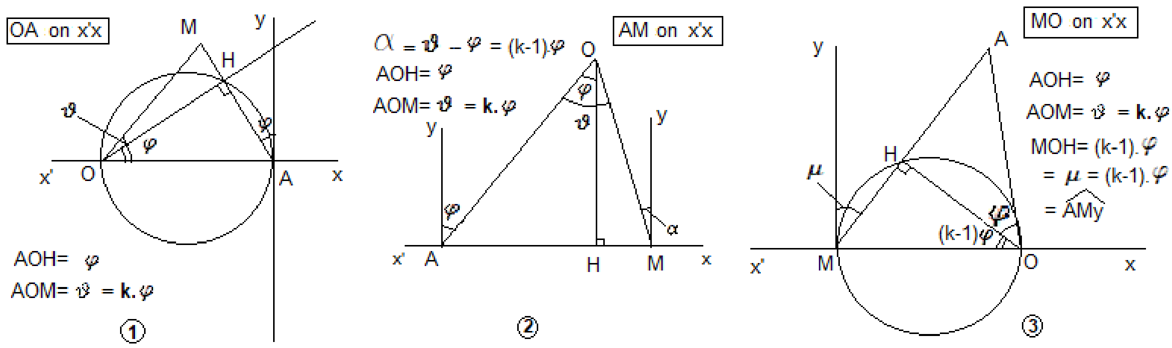

Figure 11: The 3 positions of the triangle on x'x

formula in the triangle is :

$$\frac{\sin O}{AM} = \frac{\sin A}{MO} = \frac{\sin M}{OA}$$

Here we use the two relations :

$$\widehat{AOM} = \theta = k.\phi = k.\widehat{AOH} \text{ and } \widehat{OAM} = \pi/2 - \phi$$

$$\frac{\sin\theta}{AM} = \frac{\sin(\pi/2-\phi)}{MO} = \frac{\sin(k-1).\phi}{OA}$$

6.1 1-Side OA on x'x

With the pole in O [OA on x'x] $\theta = k.\phi$:

$$\rho = OM = OA.\frac{\cos\phi}{\sin(\theta-\phi)} = OA.\frac{\cos\frac{1}{k}\theta}{\cos\frac{k-1}{k}\theta}$$

With the pole in A [OA on x'x] :
The sine formula in triangle OAM gives :

$$r = MA = OA.\frac{\sin O}{\sin\hat{M}} = OA\frac{\sin\theta}{\sin(\pi/2-(k-1)\phi)} = OA.\frac{\sin k.\phi}{\cos(k-1)\phi}$$

Another way to find this result is :

$$r = AH + HM = OA.[\sin\phi + \cos\phi.\tan(\theta-\phi)] = OA.[\sin\phi + \cos\phi.\tan(k-1)\phi]$$

$$r = OA[\frac{\sin\phi.\cos(k-1)\phi + \cos\phi.\sin(k-1)\phi}{\cos(k-1)\phi}] = OA.\frac{\sin k\phi}{\cos(k-1)\phi}$$

6.2 2-Side AM on x'x

With the pole in A [AM on x'x] :

$$\rho = AO = AM.\frac{\sin[(\pi/2-(k-1)\phi]}{\sin k.\phi} = AM.\frac{\cos(k-1)\phi}{\sin k.\phi}$$

With the pole in M [AM on x'x]then $\alpha = (k-1)\phi$:

$$r = MO = AM.\frac{\sin(\pi/2-\phi)}{\sin k\phi} = AM.\frac{\cos\phi}{\sin k\phi} = AM\frac{\cos\frac{\alpha}{k-1}}{\sin\frac{k}{k-1}\phi}$$

L'angle polaire est α count from My so :

$$\alpha = (k-1)\phi \text{ and } r = \frac{\cos\frac{1}{k-1}\alpha}{\sin\frac{k}{k-1}\phi}$$

6.3 3-Side MO on x'x

We set $\mu =$ polar angle $\widehat{AMy} = (k-1)\phi$ and $\phi = \frac{\mu}{k-1}$. And angle $A = \pi/2 - \phi$.
With the pole in M [MO on x'x] : The polar angle is $\theta = k.\phi$ and :

$$\rho = MA = MO.\frac{\sin k.\phi}{\sin(\pi/2-\phi)} = MO.\frac{\sin k.\phi}{\cos\phi} = MO\frac{\sin\frac{k}{k-1}.\mu}{\cos\frac{1}{k-1}.\mu}$$

With the pole in O [MO on x'x] :
The polar angle is $\theta = k.\phi \longrightarrow \phi = \theta/k$.

$$r = OA = MO.\frac{\cos\mu}{\sin(\pi/2-\phi)} = MO.\frac{\cos\mu}{\cos\phi} = MO.\frac{\cos(k-1).\phi}{\cos\phi}$$

$$r = MO.\frac{\cos\frac{k-1}{k}\theta}{\cos\frac{1}{k}\theta}$$

These six equations represent only three curves because each of the above equations are polar equations w.r.t. two poles on the x'x axis (respectively in O, A or M).
We can resume these results in the following table :

Side on x'x	Left-pole	Right-pole
(OA)	$OM = OA.\frac{\cos\theta/k}{\cos\frac{k-1}{k}\theta} \rightarrow (1)$	$AM = OA.\frac{\sin k.\phi}{\cos(k-1).\phi} \rightarrow (2)$
(AM)	$AO = AM.\frac{\cos(k-1)\phi}{\sin k.\phi} \rightarrow (3)$	$MO = AM.\frac{\cos\frac{1}{k-1}\alpha}{\sin\frac{k}{k-1}\alpha} \rightarrow (4)$
(MO)	$MA = MO.\frac{\sin\frac{k}{k-1}\mu}{\cos\frac{1}{k-1}\mu} \rightarrow (5)$	$OA = MO.\frac{\cos\frac{k-1}{k}\theta}{\cos\theta/k} \rightarrow (6)$

The six equations can be associated by couples of inverse curves : [(1) $\leftrightarrow$ (6)], [(2) $\leftrightarrow$ (3)] and [(4) $\leftrightarrow$ (5)].
For the case k=1/2 as example, the six equations are given in the following table :

Side on x'x	Left-pole	Right-pole
(OA)	$OM = OA.\frac{\cos 2\theta}{\cos\theta} \rightarrow (Strophoid)$	$AM = OA.\tan\phi/2 \rightarrow (Strophoid)$
(AM)	$AO = AM.\tan\phi/2 \rightarrow (Strophoid)$	$MO = AM.\frac{\cos 2.\alpha}{\sin\alpha} \rightarrow (Strophoid)$
(MO)	$MA = MO.\frac{\sin\mu}{\cos 2.\mu} \rightarrow (Eq.Hyperbola)$	$OA = MO.\frac{\cos\theta}{\cos 2.\theta} \rightarrow (Eq.Hyperbola)$

7 First case of sectrices of Maclaurin for $1 \leq m,\ n \leq 10$

Some tables of the first Sectrices of Maclaurin - when $\theta_0 = \phi_0 = 0$ - for small values of m, n are have listed at the end of this paper and show the rapid complications of these curves when the two parmeters increase. These tables use Plateau's parametric expressions.
Note that beyond m, $n > 3$ these curves are, in my knowledge, not connected to known classe of curves. And except their angular geometric definition it does seem evident to reveal new interesting geometric properties.

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(2) R.S. Luthar - Integer-sided Triangles with One Angle twice Another. College math. J. 15 (1984) 55-56.
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(4) Sectrix curve - Mathcurve.com.
(5) Sectrix of Maclaurin - Wikipedia.
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This article is the 24^{th} on plane curves.

Part I : Gregory's transformation.
Part II : Gregory's transformation Euler/Serret curves with same arc length as the circle.
Part III : A generalization of sinusoidal spirals and Ribaucour curves
Part IV: Tschirnhausen's cubic.
Part V : Closed wheels and periodic grounds
Part VI : Catalan's curve.
Part VII : Anallagmatic spirals, Pursuit curves, Hyperbolic-Tangentoid spirals, β-curves.
Part VIII : Translations, rotations, orthogonal trajectories, differential equations, Gregory's transformation.
Part IX : Curves of Duporcq - Sturmian spirals.
Part X : Intrinsically defined plane curves, periodicity and Gregory's transformation.
Part XI : Inversion, Laguerre T.S.D.R., Euler polar tangential equation and d'Ocagne axial coordinates.
Part XII : Caustics by reflection, curves of direction, rational arc length.
Part XIII : Catacaustics, caustics, curves of direction and orthogonal tangent transformation.
Part XIV : Variable epicycles, orthogonal cycloidal trajectories, envelopes of variable circles.
Part XV : Rational expressions of arc length of plane curves by tangent of multiple arc and curves of direction.
Part XVI : Logarithmic spiral, aberrancy of plane curves and conics.
Part XVII : Cesaro's curves - A generalization of cycloidals.
Part XVIII : Deltoid - Cardioid, Astroid - Nephroid, orthocycloidals
Part XIX : Tangential generation, curves as envelopes of lines or circles, arcuides, causticoides.
Part XX : Tangential dual of Steiner Habicht theorem, Circular tractrices, newtonian catenaries, circles as roulettes of a curve on a line.
Part XXI : Curves of direction, minimal surfaces and CPG duality.
Part XXII : Equality of arc length of the parabola and the Archimede spiral.A historical tale of a question that raised at the beginning of the calculus (1643 - 1668) Hobbes, Roberval, Mersenne, Torricelli, Fermat, Pascal and J. Gregory.
Part XXIII : Rectangular hyperbola - Circle Geometric properties and formal analogies.
Part XXIV : Angular relations defining curves - Sectrices of Maclaurin - Plateau's curves
Two papers in french :
1- Quand la roue ne tourne plus rond - Bulletin de l'IREM de Lille (no 15 Fevrier 1983)
2- Une generalisation de la roue - Bulletin de l'APMEP (no 364 juin 1988).
Gregory's transformation on the Web : http://christophe.masurel.free.fr

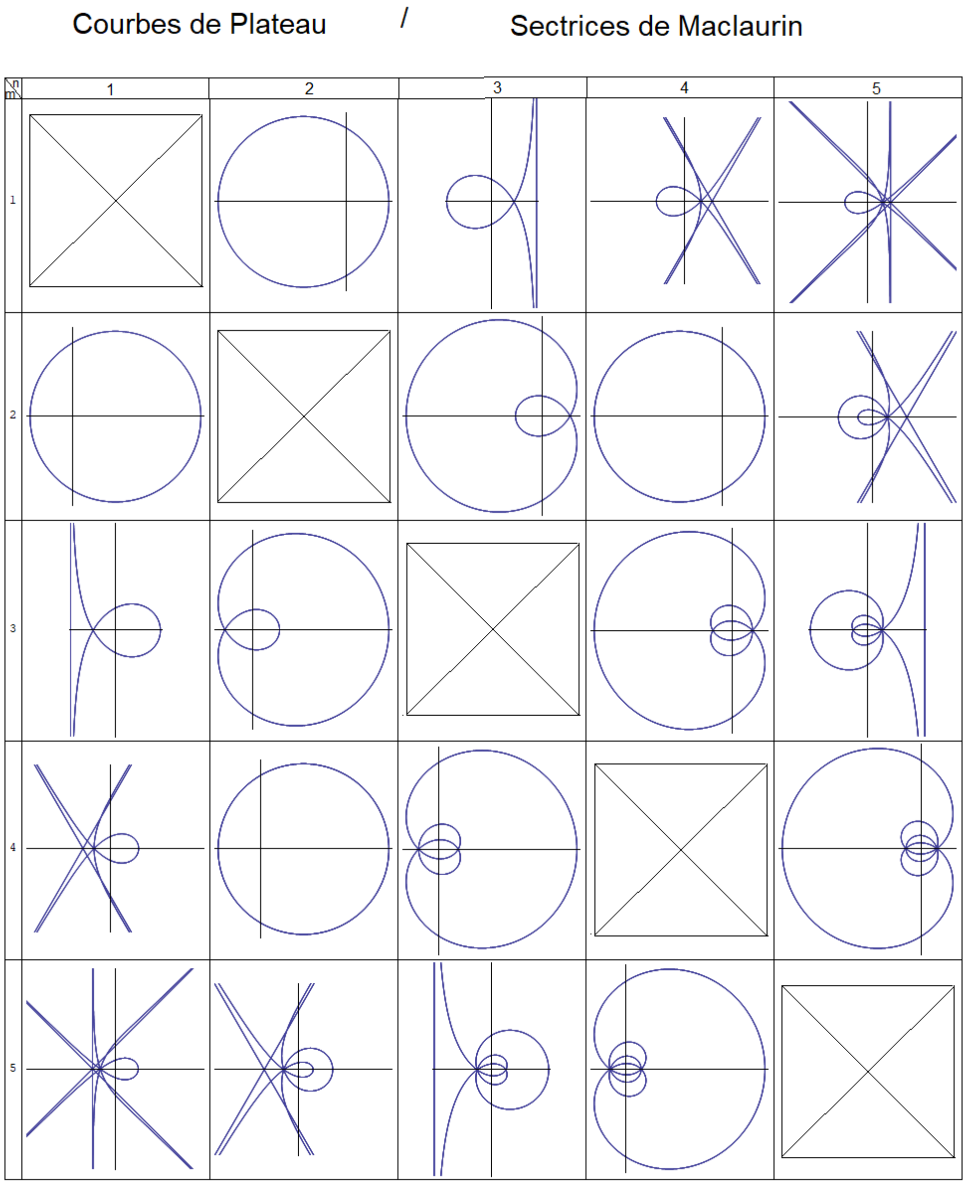

Figure 12: Sectrices of Maclaurin -Plateau's curves, $m, n \leq 5$

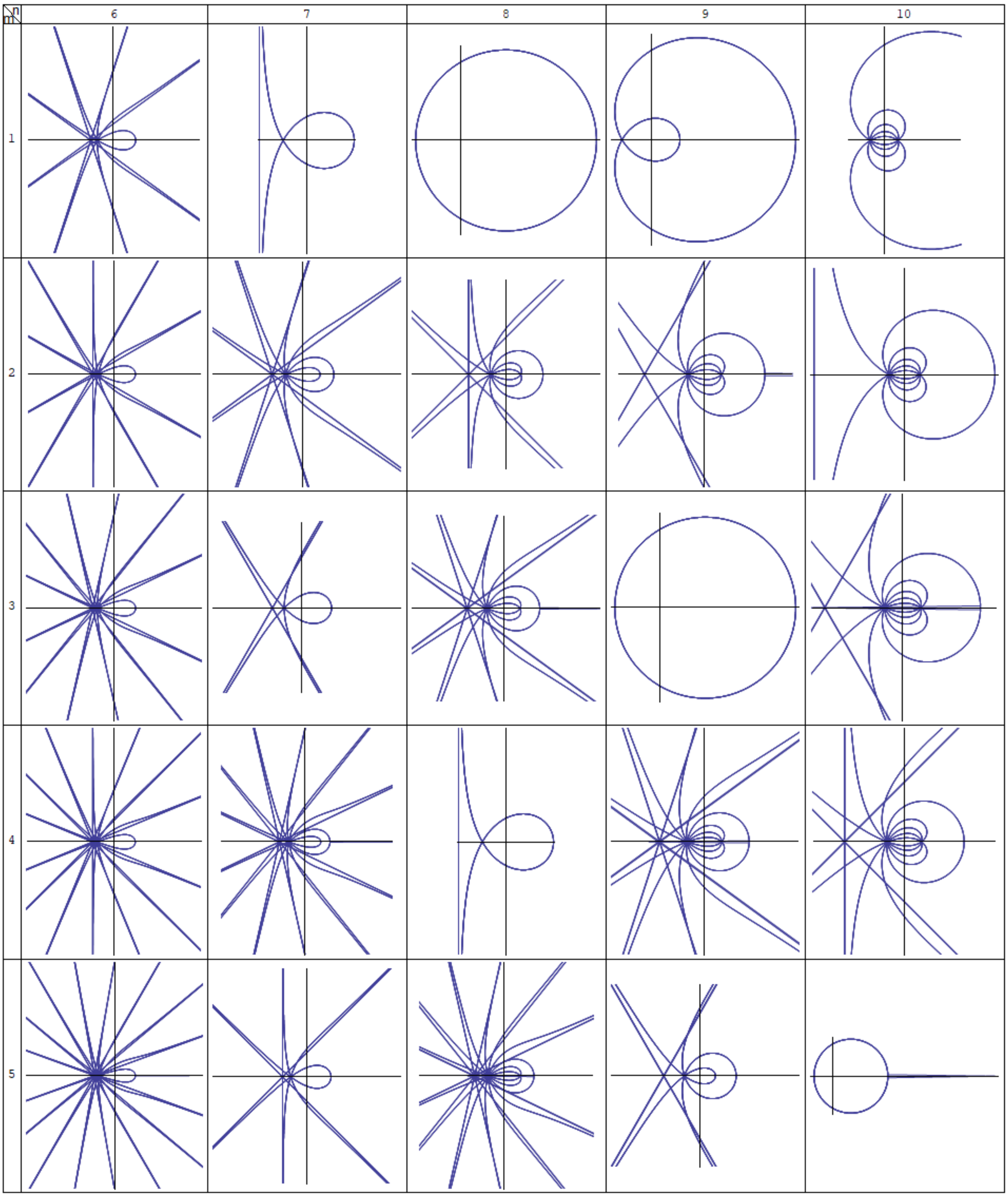

Figure 13: Sectrices of Maclaurin-Plateau's curves, $0 \leq m \leq 5$ and $6 \leq n \leq 5$

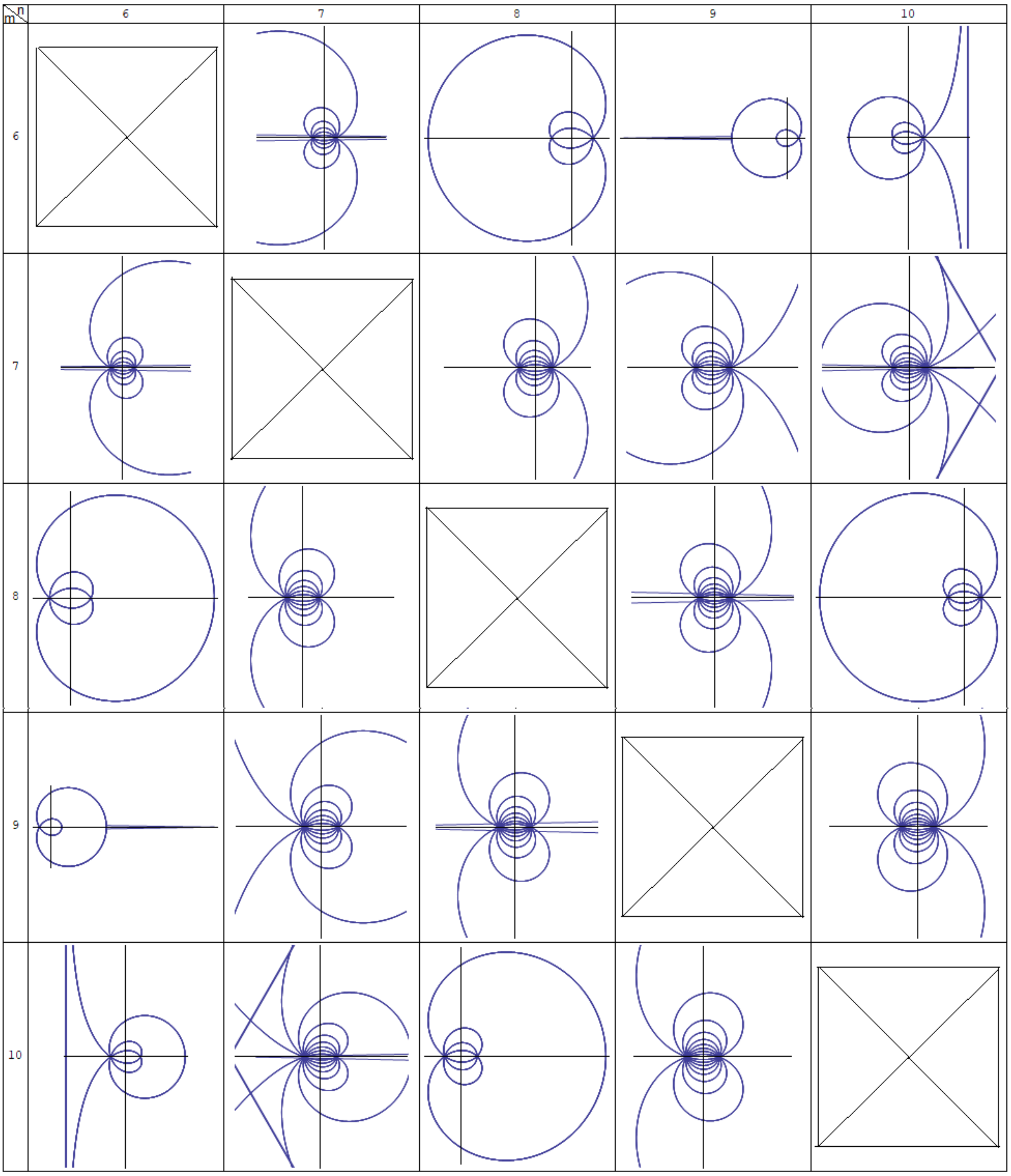

Figure 14: Sectrices of Maclaurin-Plateau's curves, $6 \leq m \leq 10$ and $6 \leq n \leq 10$

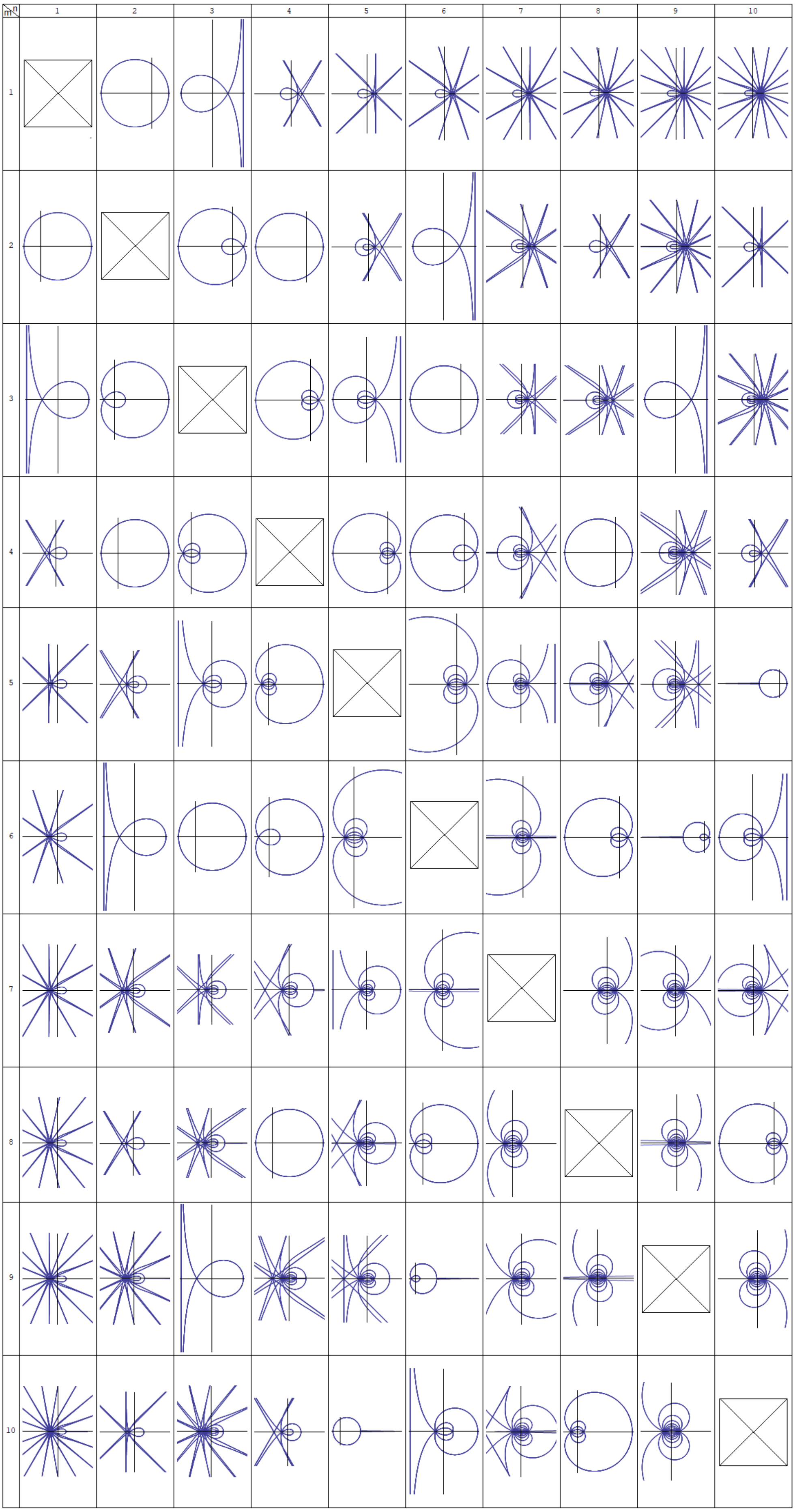

Figure 15: Sectrices of Maclaurin- Plateau's curves, $m, n < 10$